

# Physical nature of the central singularity in spherical collapse

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## Abstract

We examine here the nature of the central singularity forming in the spherically symmetric collapse of a dust cloud and it is shown that this is always a strong curvature singularity where gravitational tidal forces diverge powerfully. An important consequence is that the nature of the naked singularity forming in the dust collapse turns out to be stable against the perturbations in the initial data from which the collapse commences.

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The final outcome of the gravitational collapse of a massive matter cloud is an issue of great interest from the perspective of gravitation theory as well as its possible astrophysical implications. When there is a continual collapse without any final equilibrium for the cloud, a black hole may form where the superdense regions of the matter are hidden away from the outside observer within an event horizon of gravity or, depending on the nature of the initial data and the possible evolutions, a naked singularity could result as the end product of such a collapse.

In this context, spherical dust collapse in general relativity has been studied quite extensively. The prescription of matter as pressureless dust could perhaps be regarded as somewhat idealized. However, some authors have considered dust as a good approximation of the form of matter in the final stages of collapse<sup>1</sup>. All the same, the study of dust models over the decades has led not only to important new advances and insights into various aspects of gravitational collapse theory, but has also laid the foundation of black hole physics. The class of solutions representing an inhomogeneous spherically symmetric dust cloud was discovered and studied by Tolman, Bondi, and Lemaître<sup>2</sup>. In addition to its wide use in cosmology to model the universes which admit inhomogeneities, these models have been used extensively to investigate gravitational collapse in general relativity. For example, the special case of homogeneous dust ball was studied by Oppenheimer and Snyder<sup>3</sup>, which led to the concept and theory of black holes. This model also inspired the cosmic censorship conjecture of Penrose<sup>4</sup>, which basically states that singularities which are visible to outside observers, either locally or globally, cannot develop in gravitational collapse from regular initial data in a stable manner.

In the case of formation of a naked singularity in gravitational collapse, two aspects which become most significant are the strength of such a singularity in terms of the behavior of the gravitational tidal forces in its vicinity and its stability properties. Numerous papers on the strength have led to the classification of various subclasses within the Tolman-Bondi-Lemaître (TBL) spacetimes, where the central singularity is either shown to be weak or strong. Thus there is considerable debate on the genuineness and physicality of the central singularity in dust collapse, especially for the cases where existing literature points towards the singularity being gravitationally weak (see, e.g., Refs [5-9], and references therein). The importance of this lies in the fact that even if a naked singularity occurs rather than a black hole, if it is gravitationally weak in some suitable sense, it may not have any physical implications and it may perhaps be removable by extending the spacetime through the same<sup>10</sup>. Such, for example, would be the case for the shell-crossing naked singularities<sup>11</sup>, caused by the crossings of dust shells, through which a continuous extension of the metric has been shown to exist in certain cases, and the field equations hold in a distributional sense. Similarly, if a naked singularity is not stable in some well-defined sense, it may not have any significant physical consequences. The purpose of this paper is to investigate these aspects related to the central naked singularity forming in the dust collapse. As opposed to the shellcrossings mentioned above, the physical area of the dust shells vanish in this case when the matter shells collapse into the singularity at the center. We show here that whenever such a singularity forms as the end product of the collapse, it is always gravitationally strong.

Certain stability aspects related to naked singularities are then pointed out.

The gravitational strength of a spacetime singularity is conveniently characterized in terms of the behavior of the linearly independent Jacobi fields along the timelike or null geodesics which terminate at the singularity in the past or future<sup>12</sup>. In particular, a causal geodesic  $\gamma(s)$ , incomplete at the affine parameter value  $s = s_0$ , is said to terminate in a *strong curvature singularity* at  $s = s_0$  if the volume three-form  $V(s) = Z_1(s) \wedge Z_2(s) \wedge Z_3(s)$  (in the case of a null geodesic, this will be defined as a two-form) vanishes in the limit as  $s \rightarrow s_0$  for *all* linearly independent vorticity free Jacobi fields  $Z_1, Z_2, Z_3$  along  $\gamma(s)$ . A sufficient condition for the causal geodesic  $\gamma(s)$  to terminate in a strong curvature singularity<sup>13</sup> is that in the limit of approach to the singularity, we must have along  $\gamma$ ,

$$(s_0 - s)^2 R_{ab} V^a V^b = \text{const} > 0, \quad (1)$$

where  $V^a$  is the tangent vector to the geodesic. This expresses the requirement on the strength in terms of the rate of divergence of the Ricci tensor along the given trajectory. Essentially, the idea captured here is that in the limit of approach to such a singularity, the physical objects get crushed to a zero size, and so the idea of extension of spacetime through it would not make sense, characterizing this to be a genuine spacetime singularity (we refer to Ref. [12] for a further discussion).

We now discuss the structure and strength of the singularity forming in the TBL collapse, an issue which is all the more important when it is naked. These models are fully characterized by the initial data, which corresponds to two free functions, namely, the initial density function  $\rho(r)$  [or equivalently the mass function  $F(r)$ ] and the velocity or energy function  $f(r)$ , specified at the initial spacelike surface  $t = t_i$  at the onset of collapse. Consider the dust collapse described by the TBL metric,

$$ds^2 = -dt^2 + \frac{R'^2}{1+f} dr^2 + R^2 d\Omega^2. \quad (2)$$

The energy-momentum tensor is that of dust

$$T^{ij} = \epsilon \delta_t^i \delta_t^j, \epsilon = \epsilon(t, r) = \frac{F'}{R^2 R'}, \quad (3)$$

where  $\epsilon$  is the energy density, and  $R = R(t, r)$  is given by

$$\dot{R}^2 = f(r) + \frac{F(r)}{R}. \quad (4)$$

Here the dot and the prime denote partial derivatives with respect to the parameters  $t$  and  $r$ , respectively, and for collapse we have  $\dot{R} < 0$ . Using the remaining coordinate freedom left in the choice of rescaling of the coordinate  $r$  we set at the initial time  $t = 0$ ,

$$R(0, r) = r. \quad (5)$$

The functions  $F(r)$  and  $f(r)$  are the mass and the energy functions respectively, referred to above. We have

$$\rho(r) \equiv \epsilon(0, r) = \frac{F'}{r^2} \Rightarrow F(r) = \int \rho(r) r^2 dr. \quad (6)$$

Since at the onset of collapse at the initial surface  $t = 0$  the spacetime should be singularity free, it follows that  $F(r) = r^3 h(r)$  with  $F(0) = 0$ , and that the initial density  $\rho(r)$  must at least be a  $C^1$  function including the center at  $r = 0$ , i.e.

$$\rho(r) = \rho_c [1 - \rho_1 r^n g(r)], \quad (7)$$

where  $g(r)$  is at least  $C^0$  function with  $g(0) = 1$ ,  $n \geq 1$ , and  $\rho_1 > 0$  are constants. The term  $r^n$  is the first nonvanishing higher order term in density at  $r = 0$ . Thus, if  $\rho'(0) \neq 0$ , then  $n = 1$ . For physical reasons we take that  $\rho(0) = \rho_c \neq 0$ . The shell-focusing singularity appears at  $R = 0$  along the curve  $t = t_s(r)$  such that  $R[t_s(r), r] = 0$ . The central singularity at  $r = 0$  appears (for the case  $f = 0$ ) at the time

$$t_s = t_s(0) = \frac{2}{\sqrt{3\rho_c}}. \quad (8)$$

It is known<sup>5,8</sup> that given any initial density profile for such a cloud, one can always choose a corresponding velocity function (or vice versa) describing the infall of the matter shells, such that either a black hole or a central naked singularity would develop as desired as the outcome of the collapse. Thus a naked singularity would occur in a wide range of TBL spacetimes depending on the choice of the initial data. The structure of this naked singularity for the marginally bound, time symmetric case, when the density was taken to be an even smooth function [i.e.,  $\rho'(0) = 0$  and all the other odd derivatives of density also vanish] was analyzed analytically by Christodoulou<sup>14</sup>, and later generalized to a wider class of TBL spacetimes with the same assumption of even smooth functions by Newman<sup>6</sup>, who also showed the singularity to be gravitationally weak along the radial null geodesics terminating at the central singularity. The formation and structure of this naked singularity was then analyzed<sup>8</sup> with the only assumption of  $C^2$  differentiability of the initial mass and velocity functions. Several later papers [see, e.g., Ref. [9], and references therein] analyzed the precise role of the first three derivatives at the center of the initial density function towards determining the nature and structure of this naked singularity. The conclusion prevailing presently from all of this analysis is that in cases when the first two derivatives of the density at the center were nonvanishing, the naked singularity was gravitationally weak, because the curvature strength along the outgoing radial null geodesics did not diverge sufficiently fast. In other words, the central singularity formed in a dust collapse is naked but gravitationally weak in cases (i)  $\rho'(0) < 0$ , (ii)  $\rho'(0) = 0, \rho''(0) < 0$ , however, gravitationally strong and naked in the cases when (iii)  $\rho'(0) = \rho''(0) = 0$ , with  $\rho'''(0) < 0$  and less than a certain maximum.

Physically it is argued sometimes that the initial density distribution must be an even smooth function of  $r$ , therefore for all such cases the singularity would be weak. Furthermore, from the point of view of stability, any slight perturbation of a strong curvature singularity would involve the introduction of the term where  $\rho'(0) \neq 0, \rho''(0) \neq 0$ , and the formation of a strong curvature naked singularity would not be stable in this sense. As emphasized earlier, one could possibly remove a weak naked singularity by possibly extending the spacetime, which, however, is not possible for a strong curvature singularity. Though it has been shown<sup>9</sup> that for any given arbitrary initial density distribution one could always choose the velocity function to be such that the resulting singularity is a strong curvature naked singularity, it is possible that the measure of such profiles may be vanishingly small in the space of all initial data, and in this sense such a strong naked singularity may not be stable. Thus it could be argued that from the point of view of both the stability and curvature strength the naked singularities forming in the dust collapse may have very limited bearing on cosmic censorship hypothesis.

We shall show, however, that the structure of the central shell-focusing singularity is rather more complex than thought earlier. Considering the marginally bound case ( $f = 0$ ) for the sake of clarity, even the case of radial null geodesics can be seen to be quite involved. The geodesic equation in this case, using Eq.(2) and an integration of Eq.(4), is written as

$$\frac{dt}{dr} = \frac{\sqrt{rF} - \frac{1}{2}F't}{\sqrt{F}[r^{3/2} - \frac{3}{2}\sqrt{F}t]^{1/3}}. \quad (9)$$

It is seen that in the limit of approach to the singularity both the numerator and denominator vanish in the above equation, and therefore even in the simplest case of radial null geodesics the nature of the singularity turns out to be rather complex, which is a node for the above first order equation, giving rise to a complicated topology of integral curves near the singularity. All the analysis and related conclusions so far on this naked singularity have been based on trying to understand only some of the families of these null geodesics. Because of the complex and difficult nature of the geodesic equations, especially in the timelike case, the discussion on them has been avoided so far in the literature. It is, therefore, proposed to investigate here the nature of timelike geodesics terminating in the naked central singularity. We find that this changes the existing perception on the nature of the central naked singularity as discussed above, and in fact we get timelike radial geodesics along which the curvature growth is powerful enough near the singularity to satisfy the strong curvature condition above.

Let us consider the timelike causal curves in the TBL spacetime. Let  $U^a = (dx^a/ds)$  ( $s$  being the proper time along the trajectory) be the tangent to a timelike geodesic, satisfying  $U^a U_a = -1$ . We can express radial timelike geodesics as

$$\frac{dU^t}{dr} \pm \dot{R}' \sqrt{\frac{(U^t)^2 - 1}{1 + f}} = 0, \quad (10)$$

$$\frac{dt}{dr} = \frac{U^t}{U^r} = \pm \frac{R'}{\sqrt{1 - \frac{1}{(U^t)^2}} \sqrt{1 + f}} \Rightarrow \frac{dR}{dr} = R' \left( 1 \mp \frac{\sqrt{f + \frac{F}{R}}}{\sqrt{1 - \frac{1}{(U^t)^2}} \sqrt{1 + f}} \right), \quad (11)$$

where  $\pm$  represent outgoing and ingoing curves and we have used the variable  $R$  instead of  $t$  in the last equation. The solutions to the set of two differential equations above in the form of  $R(t, r) = f_1(r)$  and  $U^t = U^t(R, r)$ , describe the trajectories of particles following timelike geodesics.

Because of the nonlinearity, solutions to the above differential equations are not available in general. However, there is an exact solution of the above described by

$$U^a = \frac{dx^a}{ds} = \delta_t^a, \quad r = 0. \quad (12)$$

This is an ingoing radial timelike geodesic  $\gamma(s)$  which in  $(t, r)$  plane is described by  $r = 0$ , terminating at the central singularity at the coordinate time  $t = t_s(0)$ , corresponding to a particle at the center  $r = 0$  following a timelike geodesic and terminating in the singularity in future. The tangent to this timelike geodesic is given by the above equation, where  $s$  is the proper time. The equation of the trajectory is simple and is given by

$$t_s - t = s_0 - s, \quad r = 0 \quad (13)$$

where  $s_0$  is the proper time when the particle crashes into the central singularity at  $r = 0$  at the coordinate time  $t_s(0)$ .

It is now possible to show that the above is not the only geodesic terminating in the future into the singularity. In fact, there are infinitely many families of future directed null and timelike geodesics from the past of the central singularity at  $r = 0, t = t_s$ , terminating at the singularity in future. To see this, consider the past  $I^-(\gamma)$  of the timelike geodesic  $\gamma(s)$  given above. Since  $\gamma(s)$  is future endless and future incomplete, terminating in the central singularity at  $r = 0$ , the set  $I^-(\gamma)$  is a terminal indecomposable past (TIP) set as characterized by Geroch, Kronheimer and Penrose<sup>15</sup>. The boundary of this past set is then generated by null geodesics which are endless in future, terminating at the central singularity, by the theorem (2.3) of Ref. [15]. It follows that infinitely many families of null geodesics terminate at the singularity in future. If the collapse develops from the initial spacelike surface  $t = 0$ , all these null geodesics meet this surface in the past in the TBL models. Consider now any event  $p$  in  $I^-(\gamma)$ . This later set being a TIP, the past of  $p$ ,  $I^-(p)$  must be contained in  $I^-(\gamma)$ . Thus all the past directed nonspacelike curves from  $p$  must meet the initial surface  $t = 0$ , and so the set  $I^-(\gamma)$  becomes a subset of the domain of dependence (see, e.g., Ref. [16] for definitions and further details) of the initial surface  $t = 0$  in the spacetime. Then, given any event  $p$  in  $I^-(\gamma)$  and any other event  $q$  in the future of  $p$  in the same set, there exists a nonspacelike geodesic from  $p$  to  $q$ . Choosing now  $q$  to be closer and closer to the singularity, and in the limit approaching the same, we see that there are nonspacelike geodesics from  $p$  entering every neighbourhood of the central singularity,

and thus terminating in future only at the singularity. Thus, there will be infinitely many future directed nonspacelike geodesics, both null and timelike, which terminate at the central singularity in future.

One can also analyze equations (10),(11) further to understand better the structure of timelike geodesics terminating at the central singularity in future. Another such solution describing ingoing timelike causal particles is given, in the neighbourhood of the central singularity (for  $n < 3$ ), as

$$R(t, r) = X_0 r^{1+2n/3}, \quad U^t = 1 + \frac{1}{8} \left( \frac{b}{1-n/3} \right)^2 r^{2(1-\frac{n}{3})} + O(r^m), \quad (14)$$

where

$$X_0^{3/2} = \begin{cases} \frac{\rho_1}{4}, & n = 1, \\ -\frac{n\rho_1}{(3+n)(n-1)}(2 \mp \sqrt{2(3-n)}), & n \neq 1, \end{cases}$$

$$b = \left( 2 - \frac{n\rho_1}{(3+n)X_0^{3/2}} \right) \sqrt{\frac{\rho_c}{3X_0}}.$$

The trajectories described by the above equation actually represent a family in the spacetime of infinitely many ingoing radial timelike geodesics ( $\theta = \text{const}$ ,  $\phi = \text{const}$ ), terminating at the central singularity in future. In fact, the above timelike geodesics form a part of a larger family of timelike geodesic curves in the  $(t, r)$  plane that terminate in the singularity and are given in the near regions of the singularity by

$$U^t = \frac{1+C^2}{2C} \pm r^{1-n/3} \frac{1-C^2}{2C} \sqrt{\frac{\rho_c}{3X_0}} + O(r^m), \quad (15)$$

$$R(t, r) = X_0 r^{1+2n/3}, \quad (16)$$

where  $X_0^{3/2} = \frac{3\rho_1}{2(3+n)}$ ,  $C \neq 1$  is constant, and  $\pm$  represent outgoing and ingoing geodesics. Figure 1 describes a typical radial timelike geodesic terminating at the central singularity. For the case  $n \geq 3$  the singularity is already known to be strong<sup>5,9</sup>, and families of null and timelike geodesics terminate at the singularity. We have discussed the marginally bound (i.e.,  $f = 0$ ) case above for the sake of simplicity and clarity, however, for the nonmarginally bound case as well one gets similar results. As pointed out earlier, the solutions describing the particle trajectories in the  $(t, r)$  plane terminating in the singularity are not limited just to the solutions given by Eqs.(12)-(16), but there will be many more such families.

The scenario which results from the above considerations is that a single ingoing timelike geodesic given by Eq. (12), and infinitely many other radial nonspacelike geodesics always terminate in future at the central singularity. This is regardless of the form of the initial data, which in the dust collapse is in the form of the initial distribution of the density and velocities at the onset of collapse. To understand better the structure of families (outgoing or ingoing) at the singularity, we note that in case  $\rho'(0) \neq 0$  or  $\rho''(0) \neq 0$  there are always

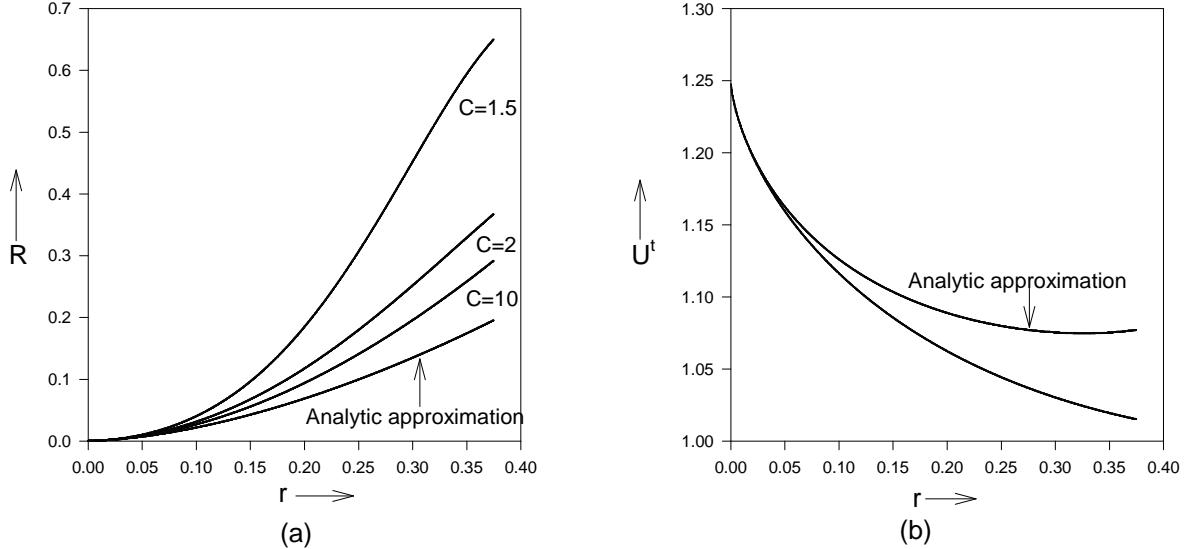


Figure 1: Typical ingoing radial timelike geodesics terminating at the central ( $r=0$ ) singularity in the TBL model. The analytic approximation is also shown. Here  $f = 0$  and  $\rho = 3 - 4.8r$ . (a) Different geodesics in the  $(r,R)$  plane. (b)  $U^t(r)$  for  $C=2$  geodesic.

families of ingoing and outgoing radial timelike geodesic curves that terminate at the naked central singularity. In cases where  $\rho'(0) = \rho''(0) = 0, \rho'''(0) \neq 0$  there is a critical value of  $\rho'''$  which determines if there would be outgoing null geodesic families; however, the ingoing family would still be there. Thus, in all cases (naked or covered), there is a nonzero measure of families of both timelike and null particles which terminate at the singularity.

In order to access the strength of the singularity, consider the quantity  $\Psi = R_{ab}U^aU^b$  along the central particle following the radial timelike geodesic  $\gamma(s)$  given by Eq. (12). The energy density throughout the spacetime is given by

$$\epsilon(t, r) = \frac{F'}{R^2 R'} = \frac{\rho(r)t_0(r)^2}{[t_0(r) - t] - \{[t_0(r) - t] + (2/3)[rt_0'(r)/t_0(r)]t\}} \quad (17)$$

Using this we get

$$\Psi = R_{ab}U^aU^b = \frac{2}{3(t_0 - t)^2} = \frac{2}{3(s_0 - s)^2}. \quad (18)$$

Since  $\Psi$  diverges as the inverse square of the proper time, it follows that the timelike geodesic  $\gamma(s)$  terminates in a strong curvature singularity. This shows, as per the criterion of strength of the singularity given above, that the central singularity forming in the dust collapse is always a strong curvature singularity. It follows, by continuity, that the curvature growth in the past of the singularity will be equally powerful in the limit of approach to the singularity along the other nonspacelike trajectories as well which meet the singularity with a tangent closer to that of the timelike geodesic  $\gamma(s)$  considered above.

A similar calculation as above for the particles following radial timelike trajectories given by Eq. (12) yields

$$\lim_{s \rightarrow s_0} ((s - s_0)^2 \Psi) = \lim_{s \rightarrow s_0} ((s - s_0)^2 R_{ab} U^a U^b) = \frac{2}{3}. \quad (19)$$

For the sake of completeness, we note that in the case of non-marginally bound dust collapse as well, where  $f \neq 0$ , the quantity  $\Psi$  along the timelike geodesics  $\gamma(s)$  given by Eqs. (12) and (14) behaves in the neighborhood of the singularity as

$$\Psi = R_{ab} U^a U^b \propto \frac{1}{(s_0 - s)^2}. \quad (20)$$

As is wellknown, the stability criteria in general relativity have still not been satisfactorily formulated. However, certain norms for stability have come to be well accepted. Amongst them is the stability of the occurrence of phenomena against small perturbations in the symmetry of the spacetime. Recent work<sup>17</sup> on the perturbations of the TBL models shows the stability of the naked singularity in this sense. Another important stability criterion for any physical phenomena, which is the result of time development of initial physical configurations (e.g., the initial densities and velocities), is the stability against perturbations of these physical parameters. This is important from the point of view of observations and measurements, and also in the context of cosmic censorship, where the role of initial data leading to a naked singularity is quite important.

To examine the stability of naked singularities from such a perspective, consider for example the marginally bound collapse scenario, in the case when for a given (at least  $C^2$  differentiable) density profile  $\rho(r)$  the singularity is naked. Consider perturbations to this initial data of the form  $\rho \rightarrow \rho + \delta\rho$ . Since any such generic infinitesimal perturbation would require the introduction of nonvanishing first or second derivatives of the density [i.e.  $\rho'(0) \neq 0$  or  $\rho''(0) \neq 0$ ; it is known<sup>18</sup> that for realistic stellar models  $\rho''(0) < 0$ ], it follows that the singularity always continues to be naked. Furthermore, as shown above, in all these cases the strong curvature condition is always satisfied regardless of the initial data. It follows that the perturbations of initial data do not alter the occurrence of a strong curvature naked singularity, which is a stable phenomena in that the singularity remains not only naked but strong also. Similarly, in nonmarginally bound collapse also, a generic perturbation of the initial data consisting of the initial density and velocity profiles will leave the naked singularity to be strong.

We thus see that the naked singularity developing in dust collapse is stable against such perturbations. What is seen is that the central singularity is always a strong curvature singularity, and infinitely many timelike geodesic trajectories terminate at this singularity, whether covered or naked. The important aspect here is the generality of this feature independent of the initial density and velocity profiles. It is of course known from earlier studies that this singularity, especially when naked, does exhibit a directional behavior for the growth of curvature in the cases such as above, and also in other collapse models such

as the Vaidya spacetimes, along the radial null geodesics families terminating at the naked singularity, in the sense that the curvature growth may be different along different families. However, even in those cases the naked singularity is always strong as per the criterion given by Krolak<sup>13</sup>. Further, in a recent development<sup>19</sup>, it has been shown that the redshift of rays from the naked singularity is finite or infinite depending upon how the density of the star decreases away from the center. Since the considerations here show the nature of the singularity to be stable and strong, the central naked singularity formed in collapse becomes a good candidate for a physical phenomena likely to have an observational signature. Though we have only considered dust here, these results seem to hold for more general equations of state as well. This will be discussed elsewhere.

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